

Uniform Convergence of Multivariate Spectral Density Estimates

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Abstract

We consider uniform moment convergence of lag-window spectral density estimates for univariate and multivariate stationary processes. Optimal rates of convergence are obtained under mild and easily verifiable conditions. Our theory complements earlier results which primarily concern weak or in-probability convergence.

1 Introduction

Consider the n -dimensional stochastic process:

$$\mathbf{Z}_t = (Z_{1t}, \dots, Z_{it}, \dots, Z_{nt})' = \mathbf{R}(\dots, \epsilon_{t-1}, \epsilon_t), \quad (1)$$

where the $b \times 1$ vectors ϵ_t are iid and $\mathbf{R}(\cdot)$ is a measurable function such that \mathbf{Z}_t exists (see Tong (1990)). Under the above conditions Z_t is strictly stationary and ergodic although existence of moments is not warranted. Note that we need not impose $n \geq b$. In fact, we are interested in nonparametric estimation, and thus issues of invertibility and related conditions are irrelevant, unlike when considering parametric estimation methods such as maximum likelihood. As a consequence of (1)

$$Z_{it} = R_i(\dots, \epsilon_{t-1}, \epsilon_t), \quad i = 1, \dots, n,$$

for a measurable scalar function $R_i(\cdot)$. In the sequel let $\mathcal{F}_t = (\dots, \epsilon_{t-1}, \epsilon_t)$.

In this paper we are interested in studying uniform convergence, in terms of distribution as well as in terms of moments, of the kernel estimator of the spectral density matrix:

$$\hat{\mathbf{f}}_T(\lambda) = \frac{1}{2\pi} \sum_{u=-T+1}^{T-1} K\left(\frac{u}{B_T}\right) e^{-iu\lambda} \mathbf{C}(u), \quad -\pi \leq \lambda < \pi, \quad (2)$$

where i denotes the complex unit and

$$\mathbf{C}(u) = \frac{1}{T} \sum^* \mathbf{Z}_t \mathbf{Z}_{t+u}' \text{ where the sum } \sum^* \text{ is for all } t, t+u \text{ between } 1 \text{ and } T,$$

B_T is the lag-window size and the kernel function satisfies

$$K(0) = 1, \text{ continuous and even, } \kappa = \int_{-\infty}^{\infty} K^2(u) du < \infty.$$

The (i, j) -entry of the spectral matrix estimator is denoted by $\hat{f}_{Tij}(\lambda)$ for every $1 \leq i, j \leq n$. Here $\hat{\mathbf{f}}_T(\lambda)$ is an estimator of the true spectral density matrix which, when exists, has the form

$$\mathbf{f}(\lambda) = \frac{1}{2\pi} \sum_{u=-\infty}^{\infty} e^{-iu\lambda} \mathbf{\Gamma}(u), \quad -\pi \leq \lambda < \pi,$$

where $\mathbf{\Gamma}(u) = \mathbb{E}(\mathbf{Z}_0 \mathbf{Z}_u')$, $u \in \mathbb{Z}$ is the autocovariance matrix satisfying $\mathbf{\Gamma}(-u) = \mathbf{\Gamma}'(u)$. Hereafter we assume $\mathbb{E}\mathbf{Z}_t = 0$ with, at minimum, bounded second moment.

To study asymptotic properties of $\hat{\mathbf{f}}_T$, we will introduce the concept of functional dependence measure. Set

$$\mathbf{Z}_{t,\{0\}} = \mathbf{R}(\dots \epsilon_{-1}, \epsilon_0^*, \epsilon_1 \dots, \epsilon_t),$$

for another iid sequence of $b \times 1$ vector ϵ_t^* , mutually independent from the ϵ_t . Define $Z_{it,\{0\}}$ accordingly. Define the m -dependent approximating sequence

$$\tilde{\mathbf{Z}}_t = \mathbb{E}(\mathbf{Z}_t | \epsilon_{t-m}, \dots, \epsilon_t) = \mathbb{E}(\mathbf{Z}_t | \mathcal{F}_{t-m,t}), \quad m \geq 0,$$

with $\mathcal{F}_{t-m,t} = \sigma(\epsilon_{t-m}, \dots, \epsilon_t)$ and \tilde{Z}_{it} accordingly. Set the p th norm, for $p > 0$, equal to:

$$\|\mathbf{Z}_t\|_p = \left(\sum_{i=1}^n \mathbb{E} |Z_{it}|^p \right)^{1/p}, \quad \|\mathbf{Z}_t\| = \|\mathbf{Z}_t\|_2.$$

For all $i = 1, \dots, n$ define the functional dependence measure

$$\delta_{t,p}^{[i]} = \|Z_{it} - Z_{it,\{0\}}\|_p,$$

and

$$\Theta_{m,p}^{[i]} = \sum_{t=m}^{\infty} \delta_{t,p}^{[i]}, \quad \Psi_{m,p}^{[i]} = \left(\sum_{t=m}^{\infty} (\delta_{t,p}^{[i]})^{p'} \right)^{\frac{1}{p'}}, \quad p' = \min(2, p),$$

$$d_{m,p}^{[i]} = \sum_{t=0}^{\infty} \min(\Psi_{m,p}^{[i]}, \delta_{t,p}^{[i]}).$$

Finally, set

$$\begin{aligned} \delta_{t,p} &= \max_{1 \leq i \leq n} \delta_{t,p}^{[i]}, \Theta_{m,p} = \max_{1 \leq i \leq n} \Theta_{m,p}^{[i]}, \\ \Psi_{m,p} &= \max_{1 \leq i \leq n} \Psi_{m,p}^{[i]}, d_{m,p} = \max_{1 \leq i \leq n} d_{m,p}^{[i]}. \end{aligned}$$

Then $\delta_{t,p}$ quantifies of dependence of \mathbf{Z}_t on ϵ_0 . Our main results in the paper need conditions on the decay of $\delta_{t,p}$.

2 Univariate case

Throughout this section assume that $Z_t, t \in \mathbb{Z}$, is a scalar stochastic process, hence $n = 1$. We also assume that $\min_{\lambda} f(\lambda) > 0$. Let $\hat{f}_T(\cdot)$ be the lag-window estimate (2) and define

$$Q(\lambda) = T[\hat{f}_T(\lambda) - \mathbb{E}\hat{f}_T(\lambda)]. \quad (3)$$

Under suitable conditions on B_T and the process (Z_t) , we have the central limit theorem

$$\frac{Q(\lambda)}{\sqrt{TB_T}} \Rightarrow N(0, \kappa f^2(\lambda)), \text{ where } \kappa = \int K^2(u) du. \quad (4)$$

For example, Anderson (1971) and Bentkus and Rudzakis (1982) dealt with linear processes and Gaussian processes, respectively and Rosenblatt (1984) considered strong mixing processes that satisfy 8th order cumulant summability conditions. Here we should consider the normalized maximum deviation

$$\max_{-\pi \leq \lambda < \pi} |Q(\lambda)|. \quad (5)$$

The following are needed on conditions on the kernel K and the lag B_T .

Assumption 1 (Condition 3 of Liu and Wu (2010)). K is an even, bounded function with bounded support in $[-1, 1]$, $\lim_{u \rightarrow 0} K(u) = K(0) = 1$, $\kappa = \int_{-1}^1 K^2(u) du < 1$ and $\sum_{l \in \mathbb{Z}} \sup_{|s-l| < 1} |K(lw) - K(sw)| = O(1)$ as $w \rightarrow 0$.

Assumption 2 (Condition 4 of Liu and Wu (2010)) There exist $0 < \underline{b} < b < 1$ and $c_1, c_2 > 0$ such that, for all large T , $c_1 T^{\underline{b}} < B_T < c_2 T^b$ holds.

Theorem 1. *Let Assumptions 1 and 2 hold. Assume $\mathbb{E}Z_0 = 0$, $\|Z_0\|_p < \infty$, $p > 4$ and*

$$\delta_{m,p} = O(\rho^m) \text{ for some } 0 < \rho < 1. \quad (6)$$

Let ν^ be such that $1 \leq \nu^* \leq p/4 - \epsilon$, some $\epsilon > 0$. Let $\lambda_l^* = \pi l/B_T$. Then*

$$\left\| \max_{0 \leq l \leq B_T} \frac{T}{B_T} \frac{|\hat{f}_T(\lambda_l^*) - \mathbb{E}[\hat{f}_T(\lambda_l^*)]|^2}{\kappa f^2(\lambda_l^*)} - 2 \log B_T + \log(\pi \log B_T) \right\|_{\nu^*} \rightarrow \|G\|_{\nu^*}, \quad (7)$$

where G denotes a Gumbel distributed random variable with cdf $e^{-e^{-x/2}}$.

Remark. Condition (6) can be weakened to

$$\begin{aligned} d_{m,p} &= O(m^{-\alpha_1}), \quad \alpha_1 > \max[1/2 - (p-4)/(2\delta p), 2\delta/p], \\ \Theta_{m,p} &= O(m^{-\alpha_2}), \quad \alpha_2 > \max[1 - (p-4)/(2\delta p), 0] \end{aligned} \quad (8)$$

where $B_T = O(T^b)$ for some $b < 1$ by Assumption 2. Thus, when the assumptions of Theorem 1 hold together with (8) and assuming $K(\cdot)$ continuous with $\hat{K}(x) = \int_{-\infty}^{\infty} e^{-i\lambda x} K(\lambda) d\lambda$ satisfying $\int_{-\infty}^{\infty} |\hat{K}(x)| dx < \infty$, then (7) holds.

Proof. By Theorem 4 and 5 of Liu and Wu (2010)

$$\mathbb{P} \left(\max_{0 \leq l \leq B_T} \frac{T}{B_T} \frac{|\hat{f}_T(\lambda_l^*) - \mathbb{E}[\hat{f}_T(\lambda_l^*)]|^2}{\kappa f^2(\lambda_l^*)} - 2 \log B_T + \log(\pi \log B_T) \leq x \right) \rightarrow e^{-e^{-x/2}}$$

under the conditions above. Uniform convergence of the moments of the maximum deviations of the spectral density estimates follows once uniform integrability of the ν^* th power of the maximum deviation is established. We now need to prove that for all ν with $1 \leq \nu < p/2$:

$$\left\| \max_{0 \leq \lambda \leq \pi} |\hat{f}_T(\lambda) - \mathbb{E}[\hat{f}_T(\lambda)]| \right\|_{\nu} = O((B_T \log B_T / T)^{1/2}). \quad (9)$$

However, this is a special case of the (multivariate) Lemma 10 reported below. QED

3 Multivariate case

Consider now the case of multidimensional \mathbf{Z}_t , with $n > 1$. We first need to derive the asymptotic distribution of the maximum deviations of the spectral density matrix estimator for $\mathbf{f}(\lambda)$. Throughout this section assume that there exists a $c_0 > 0$ such that $\mathbf{f}(\lambda) - c_0 \mathbf{I}_n$ is positive definite for all λ .

Theorem 2. (Theorem 5 of Liu and Wu (2010)) Let Assumptions 1 and 2 hold. Assume $\mathbb{E}\mathbf{Z}_0 = 0$, $\|\mathbf{Z}_0\|_p < \infty$, $p \geq 4$ and

$$\delta_{m,p} = O(\rho^m) \text{ for some } 0 < \rho < 1. \quad (10)$$

Let $\lambda_l^* = \pi|l|/B_T$. Then for all $x \in \mathbb{R}$

$$\mathbb{P} \left(\max_{0 \leq l \leq B_T} \frac{T}{B_T} \frac{|\hat{f}_{Tij}(\lambda_l^*) - \mathbb{E}[\hat{f}_{Tij}(\lambda_l^*)]|^2}{\kappa f_{ii}(\lambda_l^*) f_{jj}(\lambda_l^*)} - 2 \log B_T + \log(\pi \log B_T) \leq x \right) \rightarrow e^{-e^{-x/2}},$$

for every $i, j = 1, \dots, n$.

Proof. We generalize the proof of Theorem 5 of Liu and Wu (2010). This requires to extent a number of preliminary lemmas, presented in the Appendix. The proof then easily follows. QED

Remark. Theorem 2 holds also under the weaker condition (8).

Remark. Theorem 2 permits to evaluate simultaneous confidence intervals for any subset of elements of $\max_{0 \leq l \leq B_T} \mathbf{f}(\lambda_l^*)$ via the Bonferroni method.

Remark. Without additional difficulties, Theorems 1 and 2 of Liu and Wu (2010) can be generalized as follows:

Theorem 3. (Theorem 1 of Liu and Wu (2010)) Let Condition 1 of Liu and Wu (2010) hold. Assume $\mathbb{E}\mathbf{Z}_0 = 0$, $\|\mathbf{Z}_0\|_p < \infty$, $p \geq 2$ and $\Theta_{0,p} < \infty$. Let $1/B_T + B_T/T \rightarrow 0$. Then for every $i, j = 1, \dots, n$

$$\sup_{\lambda \in R} \|\hat{f}_{Tij}(\lambda) - f_{ij}(\lambda)\|_{p/2} \rightarrow 0.$$

Theorem 4. (Theorem 2 of Liu and Wu (2010)) Let Condition 2 of Liu and Wu (2010) hold. Assume $\mathbf{Z}_0 = 0$, $\|\mathbf{Z}_0\|_4 < \infty$ and $\Theta_{0,4} < \infty$. Let $1/B_T + B_T/T \rightarrow 0$. Then for every $i, j = 1, \dots, n$

$$\sqrt{\frac{T}{B_T}} \left(\hat{f}_{Tij}(\lambda) - \mathbb{E}[\hat{f}_{Tij}(\lambda)] \right) \rightarrow_d N(0, \omega(\lambda) \kappa f_{ii}(\lambda) f_{jj}(\lambda)),$$

for any fixed $0 \leq \lambda \leq \pi$, where $\omega(u) = 2$ if $u/\pi \in \mathbb{Z}$ and $\omega(u) = 1$ otherwise. The asymptotic distribution is complex normal for $i \neq j$.

Remark. Theorem 2 implies

$$\max_{0 \leq l \leq B_T} |\hat{f}_{Tij}(\lambda_l^*) - \mathbb{E}[\hat{f}_{Tij}(\lambda_l^*)]|^2 = O_p \left(\frac{B_T \log B_T}{T} \max_{0 \leq l \leq B_T} f_{ii}(\lambda_l^*) f_{jj}(\lambda_l^*) \right).$$

Remark. If the elements of \mathbf{Z}_t are mutually independent, the above results hold for p replaced by $p/2$.

Remark. (Remark 5 of Liu and Wu (2010)) If $K(x) - 1 = O(x)$ as $x \rightarrow 0$ and $\sum_{k \geq 1} k \delta_{k,2} < \infty$ then $\mathbb{E}\hat{f}_{Tij}(\lambda) - f_{ij}(\lambda) = O(B_T^{-1})$ and we can replace $\mathbb{E}\hat{f}_{Tij}(\lambda)$ by $f_{ij}(\lambda)$ for a sufficiently smooth model spectra, in particular whenever $T \log T = o(B_T^3)$. More in general, if $\sum_{k \geq 1} k^q \delta_{k,2} < \infty$, implying that the model spectra is q -differentiable, then $\mathbb{E}\hat{f}_{Tij}(\lambda) - f_{ij}(\lambda) = O(B_T^{-q})$. Note that under (6), it trivially holds that $\sum_{k \geq 1} k^q \delta_{k,2} < \infty$ for every $q > 1$. In this case we can replace $\mathbb{E}\hat{f}_{Tij}(\lambda)$ by $f_{ij}(\lambda)$ for a sufficiently smooth model spectra, in particular whenever $T \log T = o(B_T^{(q+1)})$. Note, however, that q will also depend on the choice of the kernel $K(\cdot)$, see Theorem 10, Chapter V, Section 4 in Hannan (1970):

$$\lim_{x \rightarrow 0} \frac{1 - K(x)}{|x|^q} = K_q < \infty.$$

As an example, $q = \infty$ for the truncated estimator but $q = 2$ for the Bartlett estimator.

Remark. We wish to have B_T as small as possible in order to achieve a quasi parametric rate but q (smoothness of the spectra) as large as possible, such that

$$T \log T = o(T^{\underline{b}(q+1)}),$$

which is satisfied if

$$\underline{b}(q+1) > 1.$$

We now present the multivariate generalization of Theorem 1.

Theorem 5. *Under the assumptions of Theorem 2, for all ν^* such that $1 \leq \nu^* \leq p/4 - \epsilon$, some $\epsilon > 0$:*

$$\left\| \max_{0 \leq l \leq B_T} \frac{T}{B_T} \frac{|\hat{f}_{Tij}(\lambda_l^*) - \mathbb{E}[\hat{f}_{Tij}(\lambda_l^*)]|^2}{\kappa f_{ii}(\lambda_l^*) f_{jj}(\lambda_l^*)} - 2 \log B_T + \log(\pi \log B_T) \right\|_{\nu^*} \rightarrow \|G\|_{\nu^*}, \quad (11)$$

for every $i, j = 1, \dots, n$, where G denotes a Gumbel distributed random variable with cdf $e^{-e^{-x/2}}$.

Proof. Convergence of the moments follows by convergence in distribution (Theorem 2) and uniform integrability of the ν -th power of $\max_{0 \leq l \leq B_T} |\hat{f}_{Tij}(\lambda_l^*) - \mathbb{E}[\hat{f}_{Tij}(\lambda_l^*)]|^2$. This is implied by uniform boundedness of the ν th moments, with $\nu^* = 2\nu - \epsilon$, which follows by Lemma 10. QED

4 Appendix

We establish here the lemmas required to proof Theorem 5.

Lemma 1. (*Lemma 1 of Liu and Wu (2010)*) Assume $\|\mathbf{Z}_t\|_p < \infty$ for $p > 1$ and $\mathbb{E}\mathbf{Z}_t = 0$. Then Lemma 1 holds for every Z_{it} , $i = 1, \dots, n$.

Proof. Trivial since each component of \mathbf{Z}_t satisfies the assumptions of Lemma 1 of Liu and Wu (2010). QED

Lemma 2. (*Proposition 1 of Liu and Wu (2010)*) Assume $\|\mathbf{Z}_t\|_{2p} < \infty$ for $p \geq 2$, $\mathbb{E}\mathbf{Z}_t = 0$ and $\Theta_{0,2p} < \infty$. Let

$$A_T^{[ij]} = \sum_{1 \leq l < l' \leq T} \alpha_{l-l'} Z_{il} Z_{jl'}, \tilde{A}_T^{[ij]} = \sum_{1 \leq l < l' \leq T} \alpha_{l-l'} \tilde{Z}_{il} \tilde{Z}_{jl'},$$

where the α_t are complex numbers. Then

$$\frac{\|A_T^{[ij]} - \mathbb{E}A_T^{[ij]} - (\tilde{A}_T^{[ij]} - \mathbb{E}\tilde{A}_T^{[ij]})\|}{T^{\frac{1}{2}} D_T \Theta_{0,2p}} \leq C_{2p} d_{m,2p} \text{ for every } i, j = 1, \dots, n,$$

setting

$$D_T = \left(\sum_{s=1}^{T-1} |\alpha_s|^2 \right)^{\frac{1}{2}}.$$

Proof. Let $E_{it-1} = \sum_{l=1}^{t-1} \alpha_{t-l} Z_{il}$, $\tilde{E}_{it-1} = \sum_{l=1}^{t-1} \alpha_{t-l} \tilde{Z}_{il}$ and

$$A_T^{[ij]*} = \sum_{1 \leq l < l' \leq T} \alpha_{l-l'} \tilde{Z}_{il} Z_{jl'} = \sum_{t=2}^T Z_{jt} \tilde{E}_{it-1}.$$

Then

$$\|\mathcal{P}_l(A_T^{[ij]} - A_T^{[ij]*})\|_p \leq I_l + II_l,$$

setting

$$\begin{aligned}
I_l &= \left\| \sum_{t=2}^T Z_{jt, \{l\}} \left[(E_{it-1} - \tilde{E}_{it-1}) - (E_{it-1, \{l\}} - \tilde{E}_{it-1, \{l\}}) \right] \right\|_p, \\
II_l &= \sum_{t=2}^T \left\| (Z_{jt} - Z_{jt, \{l\}})(E_{it-1} - \tilde{E}_{it-1}) \right\|_p.
\end{aligned}$$

Since $\|E_{it} - \tilde{E}_{it}\|_{2p} \leq C_{2p} D_T \Theta_{m+1, 2p}^{[i]}$ by Lemma 1, and $\|\tilde{Z}_{it} - \tilde{Z}_{it, \{l\}}\|_{2p} \leq \delta_{t-l, 2p}^{[i]}$ with $\sum_{t=2}^T \delta_{t-l, 2p}^{[i]} \leq \Theta_{0, 2p}^{[i]}$

$$\begin{aligned}
\sum_{l=-\infty}^T II_l^2 &\leq C_{2p}^2 D_T^2 (\Theta_{m+1, 2p}^{[i]})^2 \sum_{l=-\infty}^T \Theta_{0, 2p}^{[j]} \left(\sum_{l'=1}^{T-1} \delta_{l'-l, 2p}^{[j]} \right) \leq C_{2p}^2 D_T^2 T (\Theta_{m+1, 2p}^{[i]})^2 (\Theta_{0, 2p}^{[j]})^2. \\
\left\| \sum_{t=1}^{T-1} \left[Z_{it} - \tilde{Z}_{it} - Z_{it, \{l\}} + \tilde{Z}_{it, \{l\}} \right] \sum_{s=1+t}^T \alpha_{s-t} Z_{js, \{l\}} \right\|_p &\leq 2 \sum_{t=1}^{T-1} \min(\delta_{t-l, 2p}^{[i]}, \Psi_{m+1, 2p}^{[i]}) C_{2p} D_T \Theta_{0, 2p}^{[j]},
\end{aligned}$$

then

$$\sum_{l=-\infty}^T I_l^2 \leq C_{2p}^2 D_T^2 (\Theta_{0, 2p}^{[j]})^2 \sum_{t=-\infty}^T \Theta_{0, 2p}^{[i]} \sum_{s=1}^{T-1} \min(\delta_{s-t, 2p}^{[i]}, \Psi_{m+1, 2p}^{[i]}) \leq C_{2p}^2 D_T^2 T (\Theta_{0, 2p}^{[j]})^2 \Theta_{m+1, 2p}^{[i]} d_{m, 2p}^{[i]}.$$

Since $\Theta_{m+1, p}^{[i]} \leq d_{m, p}^{[i]}$

$$\begin{aligned}
&\left\| A_T^{[ij]} - \mathbb{E} A_T^{[ij]} - (A_T^{[ij]*} - \mathbb{E} A_T^{[ij]*}) \right\|_p^2 \leq \sum_{l=-\infty}^T \left\| \mathcal{P}_l(A_T^{[ij]} - A_T^{[ij]*}) \right\|_p^2 \\
&\leq 2C_{2p}^2 D_T^2 T (\Theta_{0, 2p}^{[j]})^2 (d_{m, 2p}^{[i]})^2 \leq 2C_{2p}^2 D_T^2 T \Theta_{0, 2p}^2 d_{m, 2p}^2.
\end{aligned}$$

The same bound applies to $\|A_T^{[ij]*} - \mathbb{E} A_T^{[ij]*} - (\tilde{A}_T^{[ij]} - \mathbb{E} \tilde{A}_T^{[ij]})\|_p^2$. QED

Lemma 3. (Proposition 2 of Liu and Wu (2010)) Assume $\mathbb{E} \mathbf{Z}_0 = 0, \|\mathbf{Z}_0\|_4 < \infty, \Theta_{0, 4} < \infty$. Let $\alpha_l = \beta_l e^{l\lambda}$ for $\lambda \in R, \beta_l \in R, 1 - T \leq l \leq T - 1, m \in N$. Define for every $i = 1, \dots, n$

$$D_l^{[i]} = A_l^{[i]} - \mathbb{E}(A_l^{[i]} | \mathcal{F}_{l-1}), \quad A_l^{[i]} = \sum_{t=0}^{\infty} \mathbb{E}(\tilde{Z}_{it+l} | \mathcal{F}_l) e^{it\lambda}$$

and

$$M_T^{[ij]} = \sum_{t=1}^T \bar{D}_t^{[i]} \sum_{l=1}^{t-1} \alpha_{l-t} D_l^{[j]}, \quad i, j = 1, \dots, n,$$

where $^-$ denotes complex conjugate. Then

$$\frac{\| \tilde{A}_T^{[ij]} - \mathbb{E} \tilde{A}_T^{[ij]} - M_T^{[ij]} \|}{m^{\frac{3}{2}} T^{\frac{1}{2}} \| Z_{i0} \|_4 \| Z_{j0} \|_4} \leq CV_m^{\frac{1}{2}}(\beta) \text{ for every } i, j = 1, \dots, n.$$

setting

$$V_m(\beta) = \max_{1-T \leq l \leq T-1} \beta_l^2 + m \sum_{l'=-1}^{-T-1} |\beta_{l'} - \beta_{l'-1}|^2.$$

Proof. Note that $A_l^{[i]} = \sum_{t=0}^m \mathbb{E}(\tilde{Z}_{it+l} | \mathcal{F}_l) e^{it\lambda}$ and that $D_l^{[i]}$ is a m -dependent martingale difference sequence. Then, setting $U_l^{[i]} = e^{i(l-t)\lambda} \mathbb{E}(A_l^{[i]} | \mathcal{F}_{l-1})$, by summation by parts:

$$\begin{aligned} \left\| \sum_{l=1}^{t-8m} \alpha_{l-t} (\tilde{Z}_{il} - D_l^{[i]}) \right\| &\leq Cm \| Z_{i0} \|_2^{\frac{1}{2}} \max_l |\beta_l| + \left\| \sum_{l=1}^{t-8m} (\beta_{l-t} - \beta_{l-t-1}) U_l^{[i]} \right\| \\ &\leq CV_m^{\frac{1}{2}}(\beta) m \| Z_{i0} \|_2. \end{aligned}$$

Likewise

$$\left\| \sum_{l=1}^{t-8m} \alpha_{l-t} (\tilde{Z}_{il} - \bar{D}_l^{[i]}) \right\| \leq CV_m^{\frac{1}{2}}(\beta) m \| Z_{i0} \|_2.$$

For $W_{1t}^{[ij]} = \tilde{Z}_{it} \sum_{l=1}^{t-8m} \beta_{l-t} e^{i(l-t)\lambda} (\tilde{Z}_{jl} - D_l^{[j]})$ then

$$\| W_{1t}^{[ij]} \| \leq CV_m^{\frac{1}{2}}(\beta) m \| Z_{i0} \|_2 \| Z_{j0} \|_2$$

yielding

$$\left\| \sum_{t=1}^T W_{1t}^{[ij]} \right\| \leq \sum_{s=1}^{4m-1} \left\| \sum_{l=0}^{(T-s)/4m} W_{1s+4ml}^{[ij]} \right\| \leq C\Delta,$$

setting $\Delta = \max_{1 \leq i, j \leq n} \Delta^{[ij]}$, $\Delta^{[ij]} = V_m^{\frac{1}{2}}(\beta) m^{\frac{3}{2}} T^{\frac{1}{2}} \| Z_{i0} \|_2 \| Z_{j0} \|_2$. Except for replacing $\| Z_{i0} \|_2 \| Z_{j0} \|_2$ with $\| Z_{i0} \|_4 \| Z_{j0} \|_4$, the same bound applies to $\left\| \sum_{t=1}^T (W_{2t}^{[ij]} - \mathbb{E} W_{2t}^{[ij]}) \right\|$ and $\left\| \sum_{t=1}^T (W_{3t}^{[ij]} - \mathbb{E} W_{3t}^{[ij]}) \right\|$ setting $W_{2t}^{[ij]} = \tilde{Z}_{it} \sum_{l=t-8m+1}^{t-1} \beta_{l-t} e^{i(l-t)\lambda} (\tilde{Z}_{jl} - D_l^{[j]})$ and $W_{3t}^{[ij]} = (\tilde{Z}_{it} - \bar{D}_t^{[i]}) \sum_{l=1}^{t-1} \beta_{l-t} e^{i(l-t)\lambda} D_l^{[j]}$. QED

Lemma 4. (Lemma 2 of Liu and Wu (2010)) Assume $\| \mathbf{Z}_t \|_p < \infty$ for $p \geq 2$ and $\mathbb{E} \mathbf{Z}_t = 0$. Then Lemma 2 holds for every Z_{it} , $i = 1, \dots, n$.

Proof. Trivial since each component of Z_t satisfies the assumptions of Lemma 2 of Liu and Wu (2010). QED

Lemma 5. (Proposition 3 of Liu and Wu (2010)) Let \mathbf{Z}_t be m -dependent with $\mathbb{E}\mathbf{Z}_t = 0, |Z_{it}| \leq M$ a.s., $m \leq T$ and $M \geq 1$. Let $S_{r,l}^{[ij]} = \sum_{t=l+1}^{l+r} Z_{it} \sum_{s=1}^{t-1} a_{T,t-s} Z_{js}$, where $l \geq 0, l+r \leq T$ and assume $\max_{1 \leq t \leq T} |a_{T,t}| \leq K_0$, $\max_{1 \leq t \leq T} \max_{1 \leq i \leq n} \mathbb{E}Z_{it}^4 \leq K_0$ for some $K_0 > 0$. Then for any $x, y \geq 1$ and $Q > 0$,

$$P(|S_{r,l}^{[ij]} - \mathbb{E}S_{r,l}^{[ij]}| \geq x) \leq 2e^{-y/4} + C_1 T^3 M^2 \left(x^{-2} y^2 m^3 (M^2 + r) \sum_{s=1}^T a_{T,s}^2 \right)^Q \\ + C_1 T^4 M^2 \max_{1 \leq i \leq n} P \left(|Z_{it}| \geq \frac{C_2 x}{y m^2 (M + r^{\frac{1}{2}})} \right) \text{ for every } i, j = 1, \dots, n.$$

Proof. Trivial since each component of \mathbf{Z}_t satisfies the assumptions of Proposition 3 of Liu and Wu (2010). QED

Lemma 6. (Theorem 6 of Liu and Wu (2010)) Let $a_{T,l} = b_{T,l} e^{i\lambda}$, where $\lambda \in R, b_{T,l} \in R$ with $b_{T,l} = b_{T,-l}$ and

$$L_T^{[ij]} = \sum_{1 \leq l, l' \leq T} a_{T,l-l'} Z_{il} Z_{jl'} \text{ and } \sigma_T^2 = \omega(\lambda) \sum_{r=1}^T \sum_{t=1}^T b_{T,t-r}^2.$$

where $\omega(u) = 2$ if $u/\pi \in Z$ and $\omega(u) = 1$ otherwise. Assume $\mathbb{E}\mathbf{Z}_t = 0, \|\mathbf{Z}_0\|_4 < \infty, \Theta_{0,4} < \infty$ and

$$\max_{0 \leq t \leq T} b_{T,t}^2 = o(\zeta_T^2), \zeta_T^2 = \sum_{t=1}^T b_{T,t}^2, \\ T\zeta_T^2 = O(\sigma_T^2), \\ \sum_{r=1}^T \sum_{t=1}^{r-1} \left| \sum_{l=1+r}^T a_{T,r-l} a_{T,t-l} \right|^2 = o(\sigma_T^4), \\ \sum_{r=1}^T |b_{T,r} - b_{T,r-1}|^2 = o(\zeta_T^2).$$

Then for $0 \leq \lambda < 2\pi$

$$\frac{(L_T^{[ij]} - \mathbb{E}L_T^{[ij]})}{\sigma_T} \rightarrow_d N(0, 4\pi^2 f_{ii}(\lambda) f_{jj}(\lambda)).$$

Proof. Note that

$$L_T^{[ij]} = A_T^{[ij]} + \bar{A}_T^{[ij]} + a_{T,0} \sum_{t=1}^T Z_{it} Z_{jt},$$

where by Lemma 1

$$\left\| \sum_{t=1}^T Z_{it} Z_{jt} - T \gamma_{ij}(0) \right\| \leq CT^{\frac{1}{2}} (\|Z_{i0}\|_4 \Theta_{0,4}^{[j]} + \|Z_{j0}\|_4 \Theta_{0,4}^{[i]}),$$

$\gamma_{ij}(0)$ denoting the (ij) entry of $\Gamma(0)$. It suffices to show that for any m

$$\frac{M_T^{[ij]} + \bar{M}_T^{[ij]}}{\sigma_T} \rightarrow_d N(0, N(0, 4\pi^2 \tilde{f}_{ii}(\lambda) \tilde{f}_{jj}(\lambda)),$$

and then use Bernstein's lemma, where

$$\tilde{f}_{ii}(\lambda) = \frac{1}{2\pi} \sum_{l=-m}^m e^{i\lambda l} \mathbb{E}(\tilde{Z}_{i0} \tilde{Z}_{il}).$$

Since $\left\| \sum_{t=1}^T \bar{D}_t^{[i]} U_t^{[j]*} \right\| \leq CT^{\frac{1}{2}} \max_{1 \leq t \leq T} |b_{T,t}|$, setting $U_t^{[i]*} = \sum_{l=(t-4m+1) \vee 1}^{t-1} a_{T,l-t} D_l^{[i]}$, we need to show that

$$\frac{1}{\sigma_T} \sum_{t=1+4m}^T (\bar{D}_t^{[i]} U_t^{[j]\diamond} + D_t^{[j]} \bar{U}_t^{[i]\diamond}) \rightarrow_d N(0, 4\pi^2 \tilde{f}_{ii}(\lambda) \tilde{f}_{jj}(\lambda)),$$

setting $U_t^{[i]\diamond} = \sum_{l=1}^{t-4m} a_{T,l-t} D_l^{[i]}$. Since $\sum_{t=1+4m}^T \|\bar{D}_t^{[i]} U_t^{[j]\diamond}\|_4^4 \leq CT \zeta_T^4 = o(\sigma_T^4)$ the Lindeberg condition conditions applies and Hall and Heyde (1980) holds if

$$\frac{1}{\sigma_T^2} \sum_{t=1+4m}^T \mathbb{E} \left(|\bar{D}_t^{[i]} U_t^{[j]\diamond} + D_t^{[j]} \bar{U}_t^{[i]\diamond}|^2 | \mathcal{F}_{t-1} \right) \rightarrow_p 4\pi^2 \tilde{f}_{ii}(\lambda) \tilde{f}_{jj}(\lambda). \quad (12)$$

Rewriting $\mathbb{E}(\cdot | \mathcal{F}_{t-1}) = \sum_{r=1}^m (\mathbb{E}(\cdot | \mathcal{F}_{t-r}) - \mathbb{E}(\cdot | \mathcal{F}_{t-r-1})) + \mathbb{E}(\cdot | \mathcal{F}_{t-m-1})$, note that for $-m \leq r \leq m-1$,

$$\begin{aligned} & \left\| \sum_{t=1+4m}^T \left(\mathbb{E}[|\bar{D}_t^{[i]} U_t^{[j]\diamond} + D_t^{[j]} \bar{U}_t^{[i]\diamond}|^2 | \mathcal{F}_{t-r}] - \mathbb{E}[|\bar{D}_t^{[i]} U_t^{[j]\diamond} + D_t^{[j]} \bar{U}_t^{[i]\diamond}|^2 | \mathcal{F}_{t-r-1}] \right) \right\|^2 \\ & \leq 4 \sum_{t=1+4m}^T \|D_t^{[i]}\|_4^4 \|U_t^{[j]\diamond}\|_4^4 \leq CT \zeta_T^4 = o(\sigma_T^4). \end{aligned}$$

Since the $D_t^{[i]}$ are $\mathcal{F}_{t-m,t}$ -measurable whilst the $U_t^{[i]\diamond}$ are \mathcal{F}_{t-4m} -measurable, $\mathbb{E}((D_t^{[i]})^2(U_t^{[j]\diamond})^2|\mathcal{F}_{t-m,t}) = (U_t^{[j]\diamond})^2\mathbb{E}((D_t^{[i]})^2)$ and (12) is equivalent to

$$\begin{aligned} & \frac{1}{\sigma_T^2} \sum_{t=1+4m}^T \left(U_t^{[i]\diamond} U_t^{[j]\diamond} \mathbb{E}(\bar{D}_t^{[i]} \bar{D}_t^{[j]}) + \bar{U}_t^{[i]\diamond} \bar{U}_t^{[j]\diamond} \mathbb{E}(D_t^{[i]} D_t^{[j]}) \right. \\ & \left. + |U_t^{[j]\diamond}|^2 \mathbb{E}(|D_t^{[i]}|^2) + |U_t^{[i]\diamond}|^2 \mathbb{E}(|D_t^{[j]}|^2) \right) \rightarrow_p \|D_t^{[i]}\|^2 \|D_t^{[j]}\|^2, \end{aligned}$$

since $\|D_t^{[i]}\|^2 = 2\pi \tilde{f}_{ii}(\lambda)$. Since $\| \sum_{t=1+4m}^T (U_t^{[i]\diamond} U_t^{[j]\diamond}) - \mathbb{E}(U_t^{[i]\diamond} U_t^{[j]\diamond}) \| = o_p(\sigma_T^2)$ and $\sum_{t=1+4m}^T |\mathbb{E}(U_t^{[i]\diamond} U_t^{[j]\diamond})| = o(\sigma_T^2)$, the result follows noticing that

$$\mathbb{E}(|U_t^{[i]\diamond}|^2) = \sum_{l=1}^{t-4m} b_{T,l-t}^2 \|D_t^{[i]}\|^2.$$

QED

Set

$$g_T^{[ij]}(\lambda) = L_T^{[ij]} - \mathbb{E}L_T^{[ij]} - \sum_{t=1}^T (Z_{it}Z_{jt} - \mathbb{E}Z_{it}Z_{jt}),$$

and

$$g_{T,m}^{[ij]}(\lambda) = \tilde{L}_T^{[ij]} - \mathbb{E}\tilde{L}_T^{[ij]} - \sum_{t=1}^T (\tilde{Z}_{it}\tilde{Z}_{jt} - \mathbb{E}\tilde{Z}_{it}\tilde{Z}_{jt}),$$

noticing that unless $i = j$ then $g_T^{[ij]}(\lambda) \neq \bar{g}_T^{[ij]}(\lambda) = g_T^{[ij]}(-\lambda)$. Set

$$\tau_T = \sqrt{TB_T}/\log B_T.$$

Lemma 7. (*Lemma 3 and Remark 7 of Liu and Wu (2010)*) *Let Assumptions 1 and 2 hold and $\mathbb{E}\mathbf{Z}_0 = 0, \|\mathbf{Z}_0\|_p < \infty, p > 4$ hold. Further assume $\delta_{s,p} = O(\rho^s)$ for some $0 < \rho < 1$. Then for any $0 < \mathcal{C} < 1$, there exists $\gamma \in (0, \mathcal{C})$ such that, for $m = \lceil T^\gamma \rceil$, for every $i, j = 1, \dots, n$*

$$\max_{1 \leq l \leq B_T} |g_T^{[ij]}(\lambda_l^*) - g_{T,m}^{[ij]}(\lambda_l^*)| = o_p(\sqrt{TB_T/\log B_T}),$$

Proof. This follows precisely Liu and Wu (2010), by setting

$$Y_{t,m}^{[ij]}(\lambda) = \tilde{Z}_{it} \sum_{s=1}^{t-1} a_{T,t-s} \tilde{Z}_{js}.$$

and $Y_{t,s_l}^{[ij]}(\lambda)$ accordingly, where $s_l = [\rho^l]$, $1 \leq l \leq r$, $r \in \mathbb{N}$ such that $0 < \rho^r < \mathcal{C}$. Also, we replace their definition of $\check{u}_r(\lambda)$ with

$$\check{u}_r(\lambda) = \sum_{t \in H_r} \left((Y_{t,s_l}^{[ij]}(\lambda) - Y_{t,s_{l+1}}^{[ij]}(\lambda)) + (Y_{t,s_l}^{[ij]}(-\lambda) - Y_{t,s_{l+1}}^{[ij]}(-\lambda)) \right).$$

QED

Remark. Lemma 4,5,6 of Liu and Wu (2010) extend without any additional difficulty.

Lemma 8. (Lemma 7 of Liu and Wu (2010)) Suppose $\mathbb{E}\mathbf{Z}_0 = 0$, $\|\mathbf{Z}_0\|_4 < \infty$ and $d_{T,4} = O((\log T)^{-2})$. For every $i, j = 1, \dots, n$ we have

(i)

$$|\mathbb{E}[g_T^{[ij]}(\lambda_1) - \mathbb{E}g_T^{[ij]}(\lambda_1)][\bar{g}_T^{[ij]}(\lambda_2) - \mathbb{E}\bar{g}_T^{[ij]}(\lambda_2)]| = O(TB_T/(\log B_T)^2)$$

uniformly on $\{(\lambda_1, \lambda_2) : 0 \leq \lambda_l \leq \pi - B_T^{-1}(\log B_T)^2, l = 1, 2 \text{ and } |\lambda_1 - \lambda_2| \geq B_T(\log B_T)^2\}$.

(ii)

$$|\mathbb{E}[g_T^{[ij]}(\lambda_1) - \mathbb{E}g_T^{[ij]}(\lambda_1)][\bar{g}_T^{[ij]}(\lambda_2) - \mathbb{E}\bar{g}_T^{[ij]}(\lambda_2)]| = O(\alpha_T TB_T \kappa f_{ii}(\lambda_1) f_{jj}(\lambda_2)),$$

uniformly on $\{(\lambda_1, \lambda_2) : B_T^{-1}(\log B_T)^2 \leq \lambda_l \leq \pi - B_T^{-1}(\log B_T)^2, l = 1, 2 \text{ and } |\lambda_1 - \lambda_2| \geq B_T^{-1}\}$ for α_T satisfying $\limsup_{T \rightarrow \infty} \alpha_T < 1$.

(iii)

$$\left| \mathbb{E}|g_T^{[ij]}(\lambda) - \mathbb{E}g_T^{[ij]}(\lambda)|^2 - 4\pi^2 TB_T f_{ii}(\lambda) f_{jj}(\lambda) \right| = O(TB_T(\log B_T)^{-2}),$$

uniformly on $\{B_T^{-1}(\log B_T)^2 \leq \lambda \leq \pi - B_T^{-1}(\log B_T)^2\}$.

Proof. (i) and (ii). Since $\|M_T^{[ij]}(\lambda) - N_T^{[ij]}(\lambda)\| = O(\sqrt{nm})$, where

$$N_T^{[ij]}(\lambda) = \sum_{t=1}^T \bar{D}_{t,\lambda}^{[i]} \sum_{l=1}^{t-1-m} \alpha_{T,l-t} D_{t,\lambda}^{[j]},$$

and $M_T^{[ij]}(\lambda) = M_T^{[ij]}$, $D_{t,\lambda}^{[j]} = \bar{D}_t^{[j]}$ as defined in Lemma 3, we need to show that

$$r_{T,\lambda_1,\lambda_2}^* = \left| \mathbb{E}(N_T^{[ij]}(\lambda_1) + \bar{N}_T^{[ij]}(\lambda_1))(\bar{N}_T^{[ij]}(\lambda_2) + N_T^{[ij]}(\lambda_2)) \right| = O(TB_T(\log B_T)^{-2}),$$

since

$$\left| \mathbb{E}(M_T^{[ij]}(\lambda_1) + \bar{M}_T^{[ij]}(\lambda_1))(\bar{M}_T^{[ij]}(\lambda_2) + M_T^{[ij]}(\lambda_2)) \right| \leq r_{T,\lambda_1,\lambda_2}^* + O(T\sqrt{mB_T} + \sqrt{Tm}B_T).$$

Easy calculations yield

$$\begin{aligned}
r_{T,\lambda_1,\lambda_2}^* = & \mathbb{E}(\bar{D}_{t,\lambda_1}^{[i]} D_{t,\lambda_2}^{[i]}) \mathbb{E}(D_{t,\lambda_1}^{[j]} \bar{D}_{t,\lambda_2}^{[j]}) \sum_{t=1}^T \sum_{l=1}^{t-m-1} K^2((t-l)/B_T) \cos((t-l)(\lambda_1 - \lambda_2)) \\
& + \mathbb{E}(\bar{D}_{t,\lambda_1}^{[i]} \bar{D}_{t,\lambda_2}^{[j]}) \mathbb{E}(D_{t,\lambda_1}^{[j]} D_{t,\lambda_2}^{[i]}) \sum_{t=1}^T \sum_{l=1}^{t-m-1} K^2((t-l)/B_T) \cos((t-l)(\lambda_1 + \lambda_2)).
\end{aligned}$$

Then follows the proof of Liu and Wu (2010).

(iii) From (i)

$$\begin{aligned}
r_{T,\lambda,\lambda}^* = & \mathbb{E}(|D_{t,\lambda}^{[i]}|^2) \mathbb{E}(|D_{t,\lambda_1}^{[j]}|^2) \sum_{t=1}^T \sum_{l=1}^{t-m-1} K^2((t-l)/B_T) \\
& + |\mathbb{E}(D_{t,\lambda}^{[i]} D_{t,\lambda}^{[j]})|^2 \sum_{t=1}^T \sum_{l=1}^{t-m-1} K^2((t-l)/B_T) \cos((t-l)(2\lambda)) \\
= & O(TB_T(\log B_T)^{-2}) + \|D_{0,\lambda}^{[i]}\|^2 \|D_{0,\lambda}^{[j]}\|^2 T \sum_{s=-B_T}^{B_T} K^2(s/B_T) \\
= & O(TB_T(\log B_T)^{-2}) + 4\pi^2 \kappa \tilde{f}_{ii}(\lambda) \tilde{f}_{jj}(\lambda),
\end{aligned}$$

where recall that $\mathbb{E}|D_{0,\lambda}^{[i]}|^2 = 2\pi \tilde{f}_{ii}(\lambda)$. QED

Lemma 9. (Lemma 8 of Liu and Wu (2010)) Set $E_T = B_T - (\log B_T)^2$. Under the conditions of Theorem 2 for every $i, j = 1, \dots, n$

$$\mathcal{P} \left(\max_{(\log B_T)^2 \leq r \leq E_T} \frac{|\sum_{l=1}^{k_T} \hat{u}_l^{[ij]}(\lambda_r^*)|^2}{4\pi^2 \kappa T B_T f_{ii}(\lambda_r) f_{jj}(\lambda_r)} - 2 \log(B_T) + \log(\pi \log B_T) \leq x \right) \rightarrow e^{-e^{-x/2}},$$

with

$$\hat{u}_l^{[ij]}(\lambda) = u_l^{[ij]}(\lambda) I\{|u_l^{[ij]}(\lambda)| \leq \sqrt{TB_T}/(\log B_T)^4\} - \mathbb{E} \left(u_l^{[ij]}(\lambda) I\{|u_l^{[ij]}(\lambda)| \leq \sqrt{TB_T}/(\log B_T)^4\} \right), 0.$$

and

$$u_l^{[ij]}(\lambda) = \sum_{t \in H_l} (\bar{Y}_{t,m}^{[ij]}(\lambda) - \mathbb{E} \bar{Y}_{t,m}^{[ij]}(\lambda)) + \sum_{t \in H_l} (\bar{Y}_{t,m}^{[ij]}(-\lambda) - \mathbb{E} \bar{Y}_{t,m}^{[ij]}(-\lambda)),$$

with

$$H_l = [(l-1)(p_T + q_T) + 1, p_T + (l-1)q_T], \quad p_T = \lceil B_T^{1+\beta} \rceil \text{ some small } \beta > 0, \quad q_T = B_T + m, \quad k_T = T/(p_T + q_T),$$

and

$$\bar{Y}_{t,m}^{[ij]}(\lambda) = \bar{Z}_{t,m}^{[i]} \sum_{s=1}^{t-1} a_{T,t-s} \bar{Z}_{s,m}^{[j]}$$

$$\bar{Z}_{s,m}^{[k]} = Z_{s,m}^{[i]'} - \mathbb{E} Z_{s,m}^{[i]'}, \quad Z_{s,m}^{[i]'} = Z_{s,m}^{[i]} I\{|Z_{s,m}^{[i]}| \leq (TB_T)^\alpha\}, \quad \alpha < 1/4.$$

Proof. This follows the proof of Lemma 8 in Liu and Wu (2010). QED

Lemma 10. *Let Assumptions 1 and 2 hold. Assume $EZ_0 = 0, \|Z\|_p < \infty, p > 4$ and*

$$\delta_{m,p}^{[i]} \leq A\rho^m \text{ for some } 0 < \rho < 1, \quad A > 0. \quad (13)$$

Then for every $1 \leq i, j \leq n$ and every $0 \leq \nu \leq p/2$, setting

$$\theta_T \equiv (TB_T \log B_T)^{\frac{1}{2}},$$

one obtains, for a constant $C_{\nu,p,b,\rho}$ that depends only on ν, p, b, ρ ,

$$\| \max_{0 \leq \lambda \leq \pi} T |\hat{f}_{Tij}(\lambda) - E[\hat{f}_{Tij}(\lambda)]| \|_\nu \leq C_{\nu,p,b,\rho} \theta_T.$$

Proof. Set $Q_{ij}(\lambda) = T |\hat{f}_{Tij}(\lambda) - E[\hat{f}_{Tij}(\lambda)]|$ for simplicity. Obviously (13) implies $\Theta_{m,p}^{[i]} \leq Am^{-\alpha}$ for any sufficiently large $\alpha > 0$. Therefore we can assume without loss of generality that α satisfies

$$b < \alpha p/2 \text{ and } (1 - 2\alpha)b < 1 - 4/p. \quad (14)$$

In fact, set $\alpha = \max(B_1, B_2) + 1$ where $B_1 = 2b/p, B_2 = 1 - (1 - 4/p)/(2b)$. In turn, (14) implies that there exists a $\beta \in (0, 1)$ such that

$$b < \alpha\beta p/2 \text{ and } (p/4 - \alpha\beta p/2)b < p/4 - 1. \quad (15)$$

In fact, β can be obtained as $\beta = \max(B_1, B_2)/\alpha + 1/2$ where $B_1/\alpha = 2b/(p\alpha), B_2/\alpha = 1/\alpha - (1 - 4/p)/(2b\alpha)$. Therefore α and β only depend on p, b .

We then follow the arguments of Theorem 10 in Xiao and Wu (2012) where, in particular, their Lemma 9 is replaced by our Lemma 2 (see Remark S.2 in Xiao and Wu (2012b))

and their Lemma 11 and 12 are generalized using our Lemmas 2, 5 and Corollary 1.6 and 1.7 of Nagaev (1979). It remains to show that their result (41) is replaced by

$$\begin{aligned}
& \left\| \sum_{t,s=1}^T c_{s,t} (Z_{it} Z_{js} - \gamma_{ij}(t-s)) \right\|_{p/2} \leq \\
& C_{p/2} \mathcal{D}_T \left(\sqrt{20} C_p \sqrt{T} \Theta_{0,p}^{[i]} \Theta_{0,p}^{[j]} + 2^{1-2/p} (\Theta_{0,p}^{[i]} \|Z_{j0}\|_p + \Theta_{0,p}^{[j]} \|Z_{i0}\|_p) \right) \\
& \leq C_{p/2} \mathcal{D}_T \left(\sqrt{20} C_p \sqrt{T} A^2 / (1-\rho)^2 + 2^{2-2/p} C_p A / (1-\rho) \right), \tag{16}
\end{aligned}$$

where $\gamma_{ij}(u)$ denotes the (ij) th entry of $\mathbf{\Gamma}(u)$ and

$$\mathcal{D}_T^2 \equiv \max \left(\max_{1 \leq s \leq T} \sum_{t=1}^T c_{s,t}^2, \max_{1 \leq t \leq T} \sum_{s=1}^T c_{s,t}^2 \right).$$

Inequality (16) is a consequence of Lemma 1 and Lemma 2, as follows. First, notice that one can rewrite

$$\sum_{t=1}^T c_{s,t} (Z_{it} Z_{js} - \gamma_{ij}(t-s)) = \sum_{t=2}^T \sum_{s=1}^{t-1} c_{s,t} (Z_{it} Z_{js} - \gamma_{ij}(t-s)) \tag{17}$$

$$\begin{aligned}
& + \sum_{s=2}^T \sum_{t=1}^{s-1} c_{s,t} (Z_{it} Z_{js} - \gamma_{ij}(t-s)) + \sum_{t=1}^T c_{t,t} (Z_{it} Z_{jt} - \gamma_{ij}(0)) \\
& \tag{18}
\end{aligned}$$

$$= A_{1T}^{[ij]} + A_{2T}^{[ij]} + \sum_{t=1}^T c_{t,t} (Z_{it} Z_{jt} - \gamma_{ij}(0)).$$

We deal with the right hand side of (17), namely $A_{1T}^{[ij]}$, the other two terms following along the same lines. For simplicity set $E_{jt-1} = \sum_{s=1}^{t-1} c_{s,t} Z_{js}$ and $D_T = (\max_{1 \leq s \leq T} \sum_{t=1}^T c_{s,t}^2)^{\frac{1}{2}}$. Then, for $\mathcal{P}_l(\cdot) \equiv E(\cdot | \mathcal{F}_l) - E(\cdot | \mathcal{F}_{l-1})$,

$$\| \mathcal{P}_l A_{1T}^{[ij]} \|_p \leq I_l + II_l,$$

setting

$$\begin{aligned}
I_l &= \left\| \sum_{t=2}^T Z_{it, \{l\}} [(E_{jt-1} - E_{jt-1, \{l\}})] \right\|_p, \\
II_l &= \sum_{t=2}^T \left\| (Z_{it} - Z_{it, \{l\}}) E_{jt-1} \right\|_p.
\end{aligned}$$

Since $\|E_{jt}\|_{2p} \leq C_{2p} D_T \Theta_{0,2p}^{[j]}$ by Lemma 1 noticing that $2p > 2$, and $\|\tilde{Z}_{it} - \tilde{Z}_{it,\{l\}}\|_{2p} \leq \delta_{t-l,2p}^{[i]}$ with $\sum_{t=2}^T \delta_{t-l,2p}^{[i]} \leq \Theta_{0,2p}^{[i]}$,

$$\sum_{l=-\infty}^T II_l^2 \leq C_{2p}^2 D_T^2 (\Theta_{0,2p}^{[j]})^2 \sum_{l=-\infty}^T \Theta_{0,2p}^{[i]} \left(\sum_{l'=1}^{T-1} \delta_{l'-l,2p}^{[i]} \right) \leq C_{2p}^2 D_T^2 T (\Theta_{0,2p}^{[i]})^2 (\Theta_{0,2p}^{[j]})^2.$$

Similarly, since

$$\left\| \sum_{t=1}^{T-1} [Z_{it} - Z_{it,\{l\}}] \sum_{s=1+t}^T c_{s,t} Z_{js,\{l\}} \right\|_p \leq 2 \sum_{t=1}^{T-1} \delta_{t-l,2p}^{[i]} C_{2p} D_T \Theta_{0,2p}^{[j]},$$

then

$$\sum_{l=-\infty}^T I_l^2 \leq 4 C_{2p}^2 D_T^2 (\Theta_{0,2p}^{[j]})^2 \sum_{t=-\infty}^T \Theta_{0,2p}^{[i]} \sum_{s=1}^{T-1} \delta_{s-t,2p}^{[i]} \leq 4 C_{2p}^2 D_T^2 T (\Theta_{0,2p}^{[j]})^2 (\Theta_{0,2p}^{[i]})^2.$$

Finally, the result follows by using $\|A_{1T}^{[ij]}\|_p^2 \leq C_p^2 \sum_{l=-\infty}^T \|\mathcal{P}_l A_{1T}^{[ij]}\|_p^2$. The same bound applies to $\|A_{2T}^{[ij]}\|_p^2$ where now D_T must be replaced by $(\max_{1 \leq t \leq T} \sum_{s=1}^T c_{s,t}^2)^{\frac{1}{2}}$. The third term follows by a straight application of Lemma 1. Hence (16) is now established.

For any $K > 1$, there exists constants $C_{p,K,\beta}$, $C_{K,\beta}$ and C_p , such that, for all $x \geq \theta_T$, we have

$$\begin{aligned} & Pr(|Q_{ij}(\lambda)| \geq x) \leq \\ & C_{p,K,\beta} x^{-p/2} (\Theta_{0,p}^{[i]} \Theta_{0,p}^{[j]})^{p/2} (L_T \log T) + C_{K,\beta} (x^{-p/2} (\Theta_{0,p}^{[i]} \Theta_{0,p}^{[j]})^{p/2} H_T)^K + e^{-C_p x^2 / (TB_T (\Theta_{0,4}^{[i]} \Theta_{0,4}^{[j]})^2)}, \end{aligned} \quad (19)$$

setting

$$\begin{aligned} L_T &\equiv (TB_T)^{\frac{p}{4}} T^{-\alpha\beta\frac{p}{2}} + TB_T^{\frac{p}{2}-1-\alpha\beta\frac{p}{2}} + T, \\ H_T &\equiv T^{1+\sqrt{\beta}(\frac{p}{4}-1)} B_T^{\frac{p}{4}}. \end{aligned}$$

Specifically, the second and the third terms in the right hand side of (19) correspond to the last two terms in inequality (44) in Xiao and Wu (2012) whereas the first term refers to the combination of theirs (50) and (51). Hence (19) follows from the generalization of inequalities (43), (44), (45) in Xiao and Wu (2012).

We shall now use the large deviation inequality (19) and conclude the proof by using $EX^a = (1/a) \int_0^\infty x^{a-1} Pr(X > x) dx$ which holds for any positive random variable X with

finite α th moment. By Theorem 7.28 in Zygmund (2002), let $Q_{ij}^* = \max_{0 \leq \lambda \leq \pi} |Q_{ij}(\lambda)|$ and $\lambda_l = \pi l / (2B)$, then $Q_{ij}^* \leq 2 \max_{0 \leq l \leq 2B} |Q_{ij}(\lambda_l)|$ since $Q_{ij}(\lambda)$ is a trigonometric polynomial with order B . Hence by (19), for a sufficiently large constant $K > 0$,

$$(20) \quad \int_{K\theta_T}^{\infty} x^{\nu-1} Pr(Q_{ij}^* \geq 2x) dx \leq (1 + 2B_T) \int_{K\theta_T}^{\infty} x^{\nu-1} \max_{\lambda} Pr(|Q_{ij}(\lambda)| \geq x) dx$$

$$\leq C_{p,K,\beta,\nu} (1 + 2B_T) (\theta_T^{\nu-p/2} (\Theta_{0,p}^{[i]} \Theta_{0,p}^{[j]})^{p/2} L_T \log T + ((\Theta_{0,p}^{[i]} \Theta_{0,p}^{[j]})^{p/2} K^{-p/2} H_T)^K \theta_T^{\nu-pK/2} + \theta_T^{\nu} B_T^{-C_{p,\nu} K^2 / (\Theta_{0,4}^{[i]} \Theta_{0,4}^{[j]})^2}).$$

Elementary calculations show that, under (15), the right hand side of (20) is $O(\theta_T^{\nu})$ if we choose a large enough K . Hence we have $\|Q_{ij}^*\|_{\nu} = O(\theta_T)$ since $\int_0^{K\theta_T} x^{\nu-1} Pr(Q_{ij}^* \geq 2x) dx \leq (K\theta_T)^{\nu} / \nu$. In particular the two inequalities in (15) allow to bound the terms associated with the first and the second component of L_T . The last term of L_T does not require any restrictions since $p/4 > 1$. The term involving H_T requires K large enough such that

$$A_1 = \frac{b}{(p/4 - 1)(1 - \sqrt{\beta})} < K, \quad (21)$$

and the third, last, term on the right hand side of (20) requires K large enough such that

$$\left(\frac{(\Theta_{0,4}^{[i]} \Theta_{0,4}^{[j]})^2}{C_{p,\nu}} \right)^{\frac{1}{2}} < K. \quad (22)$$

Since $\Theta_{0,p}^{[i]} = \sum_{t=0}^{\infty} \delta_{t,p}^{[i]} \leq A/(1 - \rho)$ for every $i = 1, \dots, n$, it follows that (22) is implied by

$$A_2 = \frac{A^2}{C_{p,\nu}^{\frac{1}{2}} (1 - \rho)^2} < K.$$

Then set $K = \max(A_1, A_2) + 1$. This implies that K only depends on ν, p, b, ρ . Since the same applies to α and β , it follows that we can construct a constant $C_{\nu,p,b,\rho}$ that satisfies our statement. QED

Remark. Lemma 10 can be extended to the case when $\delta_{m,p}^{[i]} = O(m^{-\alpha_i})$, for some $\alpha_i > 0$, by suitable modification of (14), (15), (21) and (22).

5 References

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